TRANSVERSE VIBRATIONS OF TRIANGULAR MEMBRANES

BY B. R. SETH

(*From the Department of Mathematics, Hindu College, Delhi*)

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THE solutions for the transverse vibrations of circular and rectangular membranes are well known. The case of a right-angled isosceles triangular membrane can be as easily solved as that of a rectangular one,[1] and that of an equilateral triangular membrane has been discussed by Lamé who finds that the frequency, $(p/2\pi)$, is given by

$$\frac{p}{2\pi}=\frac{c}{a}(m^2+n^2+mn)^{\frac{1}{2}}, \ [m, n \text{ integers}] \tag{1}$$

a being the height of the triangle and c the ratio of the uniform tension T_1 and the superficial density ρ. If z be the displacement, Routh[2] has given in terms of trilinear co-ordinates the particular solution obtained from (1) by putting m or $n=0$ as

$$z=\sin\frac{n\pi\alpha}{a}\sin\frac{n\pi\beta}{a}\sin\frac{n\pi\gamma}{a}\cos pt, \tag{2}$$

α, β, γ being the trilinear co-ordinates. This particular solution has also been recently obtained by D. G. Christopherson.[3]

As Lamé's Leçons Sur L' Elasticite in which the general solution (1) is given is not readily available, it has been independently obtained in this paper. New results have been obtained for:

(*a*) an isosceles triangle containing an angle of 120°;

(*b*) a right-angled triangle containing an angle of 60°.

If x, y, z are the co-ordinates of a point of the membrane in the displaced position, the xy plane coinciding with the equilibrium position, we know that for small displacements z should satisfy

$$\frac{\partial^2 z}{\partial t^2}=c^2\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right). \tag{3}$$

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[1] Rayleigh, *Theory of Sound*, I, 1894, pp. 317–18.

[2] Routh, *Advanced Rigid Dynamics*, 1905, p. 457.

[3] D. G. Christopherson, *Quart. J. of Math.*, 1940, **11**, p. 65.

If we assume the time factor in z to be of the form A cos $(pt+\epsilon)$, (3) becomes

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}+\frac{p^2}{c^2}z=0. \tag{3·1}$$

The solution of (3·1) may be written in the form

$$z=\frac{\sin}{\cos}\left(\frac{p\cos\theta}{c}x\right)\frac{\sin}{\cos}\left(\frac{p\sin\theta}{c}y\right). \tag{4}$$

Isosceles Triangular Membranes

Let the vertical angle of the triangle be 2α. Let us take the origin at the vertex and the axes of x and y perpendicular and parallel to the base of the triangle. The sides of the triangle are $x=a$, $y=\pm x\tan\alpha$. One form of z satisfying the boundary condition $z=0$ is

$$z=\cos\frac{(2m+1)\pi x}{2a}\cos\frac{(2n+1)\pi y}{2a\tan\alpha}-\cos\frac{(2n+1)\pi x}{2a}\cos\frac{(2m+1)\pi y}{2a\tan\alpha}, \tag{5}$$

m and n being integers. This satisfies (3) only when $\alpha=\frac{1}{4}\pi$, and then we have

$$p^2=\frac{c^2\pi^2}{4a^2}\{(2m+1)^2+(2n+1)^2\}, \tag{6}$$

the gravest mode being given by $m=0, n=1$, that is,

$$\frac{p}{2\pi}=\frac{c}{2a}\sqrt{2\cdot 5}=(0\cdot 791)\frac{c}{a}. \tag{7}$$

This solution gives the asymmetrical vibrations of a right-angled isosceles membrane. The symmetrical are given by

$$z=\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a}-\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}, \tag{8}$$

which obviously makes $y=0$ a nodal line.

Equilateral membrane.—If we put

$$z=2\sin\frac{(m-n)\pi x}{a}\cos\frac{(m+n)\pi y}{a\tan\alpha}$$
$$-2\sin\frac{(2m+n)\pi x}{a}\cos\frac{n\pi y}{a\tan\alpha}+2\sin\frac{(2n+m)\pi x}{a}\cos\frac{m\pi y}{a\tan\alpha}, \tag{9}$$

m, n being integers, we see that the boundary condition is satisfied for any isosceles triangle. But it exactly satisfies (3) only when $\alpha=\frac{1}{6}\pi$, and in that case[4]

$$\frac{p}{2\pi}=\frac{c}{a}(m^2+n^2+mn)^{\frac{1}{2}}. \tag{10}$$

[4] This solution also holds good for the symmetrical vibrations of a rhombus of angle 120° with the shorter diagonal as a nodal line.

This gives the asymmetrical vibrations of an equilateral membrane. The gravest mode is given by

$$z = 2 \sin \frac{2\pi x}{a} - 4 \sin \frac{\pi x}{a} \cos \frac{\pi y \sqrt{3}}{a}, \tag{11·1}$$

$$\frac{p}{2\pi} = \frac{c}{a}. \tag{11·2}$$

If we put $n = 0$ in (9), we get the result to which (2) reduces when changed into Cartesian co-ordinates, and which has been also obtained by Christopherson.

The symmetrical modes are given by

$$\begin{aligned} z = {} & 2 \sin \frac{(m-n)\pi x}{a} \sin \frac{(m+n)\pi y\sqrt{3}}{a} \\ & + 2 \sin \frac{(2m+n)\pi x}{a} \sin \frac{n\pi y\sqrt{3}}{a} \\ & - 2 \sin \frac{(2n+m)\pi x}{a} \sin \frac{m\pi y\sqrt{3}}{a}. \end{aligned} \tag{12}$$

All the three medians are now nodal lines.

Isosceles Triangle Containing an Angle of 120°

In this case we put

$$\begin{aligned} z = {} & 2 \sin \frac{(m-n)\pi x}{a} \sin \frac{(m+n+1)\pi y \tan \alpha}{a} \\ & - 2 \cos \left(2m+1+\frac{2n+1}{2}\right) \frac{\pi x}{a} \cos \frac{(2n+1)\pi y \tan a}{2a} \\ & + 2 \cos \left(2n+1+\frac{2m+1}{2}\right) \frac{\pi x}{a} \cos \frac{(2m+1)\pi y \tan \alpha}{2a}. \end{aligned} \tag{13}$$

This makes $z = 0$ on the boundary and satisfies (3) if $\alpha = \frac{1}{3}\pi$ and the sides are given by $x = a$, $y = x\sqrt{3}$, $y + x\sqrt{3} = 2a/\sqrt{3}$. The general expression for the frequency is now given by

$$\frac{p}{2\pi} = \frac{1}{2}\frac{c}{a}\{(2m+1)^2 + (2n+1)^2 + (2m+1)(2n+1)\}^{\frac{1}{2}}. \tag{14}$$

In the gravest mode we have

$$\begin{aligned} z = {} & 2 \sin \frac{2\pi x}{a} \sin \frac{\pi y\sqrt{3}}{a} - 2 \cos \frac{5\pi x}{2a} \cos \frac{\pi y\sqrt{3}}{2a} \\ & + 2 \cos \frac{\pi x}{2a} \cos \frac{3\pi y\sqrt{3}}{2a}, \end{aligned} \tag{15·1}$$

$$\frac{p}{2\pi} = \frac{1}{2}\sqrt{7}\,\frac{c}{a} = (1·323)\,\frac{c}{a} = (0·882)\,\frac{c}{h}. \tag{15·2}$$

These give the symmetrical vibrations of an isosceles triangular membrane of height h containing an angle of 120°. $y = a/\sqrt{3}$ is a nodal line and the gravest mode appears to be symmetrical.

Right-angled Triangle Containing an Angle of 60°

In this case we can take the sides as $y = a/\sqrt{3}$, $x = a$, $y = x\sqrt{3}$. The result for this case follows at once from the symmetrical vibrations of an isosceles triangle containing an angle of 120°. We easily see that z is of the same form as given in (13). The lowest tone is given by (15). The general form of the frequency is the same as given in (14).

It has not been found possible to get an exact solution for any other case excepting the four discussed above. In all other cases an approximate value of the gravest mode may be obtained by using Rayleigh's method when the normal types cannnot be accurately determined.